A new method for treating supersonic flow past nearly plane wings

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Function-theoretic techniques are used for determining the supersonic flow past a wing of infinite length having a straight leading edge. The problem is reduced to a set of dual integral equations which are solved by function-theoretic methods. This approach is shown to be useful for treating more general-type leading edges yielding closed-form solutions which heretofore have been unobtainable. The general solutions obtained here are shown to be in complete agreement with previous solutions for the flow at large distances downstream on the aerofoil. For a wing of infinite span, the solution reduces to the well-known perturbation potential for a distribution of sources in a horizontal plane.

1. Introduction

The linearized theory for supersonic flow past an aerofoil of zero aspect ratio was considered by Stewartson (1950) and later by Lomax, Heaslet & Fuller (1951). The former obtained the asymptotic value of the perturbation potential on a plane wing for large distances downstream. In the latter paper, the authors reduced the problem to a set of Volterra integral equations of the second kind for the loading coefficient on the centre-line of the wing. The solution of these equations can be determined by numerical processes. Gunn (1947) showed that this problem can be reduced to a set of dual integral equations by operational techniques. He did not pursue this method further, however, since he believed that its direct solution was unobtainable.

In the present paper, we obtain a general solution for the perturbation potential for supersonic flow over a nearly planar wing of infinite length having either a purely supersonic leading edge or a partially supersonic and partially subsonic leading edge. By the use of function-theoretic techniques, we show that the dual integral equations mentioned above are, indeed, solvable and lead to a closed-form solution for the perturbation potential.

It is shown that for large distances downstream on the aerofoil the solution for the perturbation potential agrees with an approximate solution obtained by Stewartson (1950). It is further demonstrated that as the span of the aerofoil becomes infinite, the potential function reduces to the well-known potential solution due to a distribution of sources in a semi-infinite horizontal plane.

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2. Mathematical formulation of the dual integral equations

In the analysis which follows, it is assumed that the flow is inviscid and isentropic. The perturbation of a uniform high-speed flow caused by a thin wing of infinite length and finite span is assumed so small that the squares and higher powers of the perturbation velocity components and their derivatives can be neglected in the equations of motion of the fluid and in the boundary conditions. A linear partial differential equation is thus obtained for the first-order perturbation velocity potential.

The wing is situated in a right-handed Cartesian co-ordinate system with the direction of the supersonic stream taken along the positive y-axis. The wing has a semi-span b along the x-axis with its centre-line along the y-axis. The mean surface of the wing is denoted by S and lies in the $z = 0$ plane, $y > 0$, $|x| < b$. The remaining surface of the $z = 0$ plane is denoted by R.

Let U denote the velocity of the undisturbed free stream and let M denote the Mach number. The perturbation potential ϕ satisfies the partial differential equation governing the first-order perturbation velocity

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial z^2} = \alpha^2\frac{\partial^2\phi}{\partial y^2}, \tag{1}$$

where $\alpha = (M^2-1)^{\frac{1}{2}}$.

The perturbation potential may be considered as the superposition of two independent solutions, viz. the symmetrical and the anti-symmetrical parts. Each of these is considered separately. The symmetrical problem corresponds to a wing having a small, but finite thickness and set at a zero angle of incidence with the free stream. The anti-symmetrical problem corresponds to a wing of infinitesimal thickness and may possess a non-zero lifting force. Since the wing is taken as infinite in length and parallel to the free-stream direction, it has no trailing vortex. The wing is assumed to have a straight leading edge along the x-axis, $|x| < b$. A more general-type leading edge is treated subsequently.

Set $\phi = \phi_1+\phi_2$, where ϕ_1 is the symmetric and ϕ_2 the antisymmetric function with respect to the $z = 0$ plane, i.e.

$$\left.\begin{aligned} \phi_1(x,y,z) &= \tfrac{1}{2}\{\phi(x,y,z)+\phi(x,y,-z)\} \\ \text{and}\qquad \phi_2(x,y,z) &= \tfrac{1}{2}\{\phi(x,y,z)-\phi(x,y,-z)\}. \end{aligned}\right\} \tag{2}$$

The derivative of $\phi_1(x,y,z)$ normal to the $z = 0$ plane is continuous on R and discontinuous on the wing surface S. The boundary conditions which must be satisfied by $\phi_1(x,y,z)$ on the surface $z = 0^+$ are

$$2\frac{\partial\phi_1}{\partial z} = \lim_{z\to 0^+}\frac{\partial\phi}{\partial z}(x,y,z)-\lim_{z\to 0^-}\frac{\partial\phi}{\partial z}(x,y,z) \quad \text{on } S. \tag{3a}$$

$$\partial\phi_1/\partial z = 0 \quad \text{on } R, \tag{3b}$$

$$\phi_1 \text{ is continuous everywhere in } z \geqslant 0^+. \tag{3c}$$

The boundary conditions which $\phi_2(x,y,z)$ must satisfy are

$$2\frac{\partial\phi_2}{\partial z} = \lim_{z\to 0^+}\frac{\partial\phi}{\partial z}(x,y,z)-\lim_{z\to 0^-}\frac{\partial\phi}{\partial z}(x,y,z) \quad \text{on } S, \tag{4a}$$

$$\phi_2 = 0 \quad \text{on } R, \tag{4b}$$

$$\phi_2 \text{ is continuous everywhere in } z \geqslant 0^+. \tag{4c}$$

$\partial\phi/\partial z(x,y,z)$ is proportional to the slope of the aerofoil for $z \to \pm 0$ and is prescribed on S.

The symmetrical potential function ϕ_1, satisfying (1) and the boundary conditions (3), has been treated by Puckett (1946) and Ward (1955). The solution to this problem is given by

$$\phi_1(x,y,z) = -\frac{1}{\pi}\iint \frac{\partial\phi}{\partial z'}(x',y',0^+)\frac{dy'\,dx'}{\{(y-y')^2-\alpha^2(x-x')^2-\alpha^2z^2\}^{\frac{1}{2}}}, \tag{5}$$

where the integration is taken over that portion of the $z=0$ plane for which

$$y' \leqslant y-\alpha\{(x-x')^2+z^2\}^{\frac{1}{2}}. \tag{6}$$

The antisymmetric problem can be solved employing operational methods. For this purpose we introduce the Laplace transform of ϕ_2

$$\Phi_2(x,p,z) = \int_0^\infty e^{-py}\,\phi_2(x,y,z)\,dy. \tag{7}$$

Since the perturbation functions $\phi_2(x,0,z)$, $\partial\phi_2/\partial y(x,0,z)$ are zero, the partial differential equation satisfied by ϕ_2,

$$\frac{\partial^2\phi_2}{\partial x^2}+\frac{\partial^2\phi_2}{\partial z^2} = \alpha^2\frac{\partial^2\phi_2}{\partial y^2},$$

is transformed as

$$\frac{\partial^2\Phi_2}{\partial x^2}+\frac{\partial^2\Phi_2}{\partial z^2} = \alpha^2p^2\Phi_2. \tag{8}$$

The transformed boundary conditions (4) are

$$\left.\begin{aligned}\frac{\partial\Phi_2}{\partial z}(x,p,0) &= \int_0^\infty e^{-py}\frac{\partial\phi_2}{\partial z}(x,y,0)\,dy, \quad 0<|x|<b \quad \text{(prescribed)},\\ \Phi_2(x,p,0) &= \int_0^\infty e^{-py}\,\phi_2(x,y,0)\,dy = 0, \quad b\leqslant|x|<\infty.\end{aligned}\right\} \tag{9}$$

Set

$$\phi_2(x,y,z) = \phi_e(x,y,z)+\phi_0(x,y,z), \tag{10}$$

where

$$\left.\begin{aligned}\phi_e(x,y,z) &= \tfrac{1}{2}\{\phi_2(x,y,z)+\phi_2(-x,y,z)\},\\ \phi_0(x,y,z) &= \tfrac{1}{2}\{\phi_2(x,y,z)-\phi_2(-x,y,z)\}.\end{aligned}\right\} \tag{11}$$

Denote the Laplace transform of $\phi_e(x,y,z)$ and $\phi_0(x,y,z)$ by Φ_e and Φ_0, respectively. Assume solutions for Φ_e and Φ_0 of the form

$$\Phi_e(x,p,z) = \int_0^\infty a_e(q,p)\cos(\alpha qx)\exp\{-\alpha z(p^2+q^2)^{\frac{1}{2}}\}\,dq, \tag{12a}$$

$$\Phi_0(x,p,z) = \int_0^\infty a_0(q,p)\sin(\alpha qx)\exp\{-\alpha z(p^2+q^2)^{\frac{1}{2}}\}\,dq, \tag{12b}$$

where $a_e(q,p)$ and $a_0(q,p)$ are to be found. It is easily verified that (12) formally satisfies (8). Both the even and odd characteristics of $\Phi_e(x,p,z)$ and $\Phi_0(x,p,z)$, respectively, are preserved in (12). The functions $a_e(q,p)$ and $a_0(q,p)$ are determined from the transformed boundary conditions (9). We assume that differentiation under the integral sign is valid. These boundary conditions may then be expressed as two sets of dual integral equations

$$\left.\begin{aligned}\frac{\partial\Phi_e}{\partial z}(x,p,0) &= -\alpha\int_0^\infty a_e(q,p)\cos(\alpha qx)(q^2+p^2)^{\frac{1}{2}}\,dq \quad (0<x<b),\\ 0 &= \int_0^\infty a_e(q,p)\cos(\alpha qx)\,dq \quad (b<x<\infty);\end{aligned}\right\} \tag{13}$$

$$\left.\begin{aligned}\frac{\partial\Phi_0}{\partial z}(x,p,0) &= -\alpha\int_0^\infty a_0(q,p)\sin(\alpha qx)(q^2+p^2)^{\frac{1}{2}}\,dq \quad (0<x<b);\\ 0 &= \int_0^\infty a_0(q,p)\sin(\alpha qx)\,dq \quad (b<x<\infty).\end{aligned}\right\} \tag{14}$$

The functions $\partial\Phi_e(x,p,0)/\partial z$ and $\partial\Phi_0(x,p,0)/\partial z$ are known in the interval $0<x<b$ and may be expressed in terms of $\partial\Phi_2/\partial z(x,p,0)$.

3. Solution of the dual integral equations

The dual integral equations

$$\left.\begin{aligned}&\int_0^\infty a_e(q,p)\cos(\alpha qx)(q^2+p^2)^{\frac{1}{2}}\,dq = -\frac{1}{\alpha}\frac{\partial\Phi_e}{\partial z}(x,p,0) \quad (0<x<b),\\ &\int_0^\infty a_e(q,p)\cos(\alpha qx)\,dq = 0 \quad (b<x<\infty),\end{aligned}\right\} \tag{15}$$

can be solved using function-theoretic methods, i.e. by reducing the above equations to a set of functional equations between analytic functions.

By a simple change of variables, (15) can be written as

$$\int_0^\infty rA(r)\cos(rt)\,dr = g(t) \quad (0<t<1), \tag{16a}$$

$$\int_0^\infty \frac{rA(r)}{(r^2+s^2)^{\frac{1}{2}}}\cos(rt)\,dr = 0 \quad (1<t<\infty), \tag{16b}$$

where
$$\left.\begin{aligned}a_e\left(\frac{r}{\alpha b},\frac{s}{\alpha b}\right) = \frac{rA(r)}{(r^2+s^2)^{\frac{1}{2}}},\quad g(t) = -\alpha b^2\frac{\partial\Phi_e}{\partial z}\left(tb,\frac{s}{\alpha b},0\right),\\ x = tb,\quad s=\alpha bp.\end{aligned}\right\} \tag{17}$$

Assuming continuity of the integrands in (16) with respect to r and t and uniform convergence of these integrals with respect to t, one may integrate the set of dual integral equations with respect to t

$$\int_0^\infty A(r)\sin xr\,dr = \int_0^x g(t)\,dt \quad (0<x<1); \tag{18a}$$

$$\int_0^\infty \frac{A(r)}{(r^2+s^2)^{\frac{1}{2}}}\sin xr\,dr = \int_0^\infty \frac{A(r)\sin r}{(r^2+s^2)^{\frac{1}{2}}}\,dr \quad (1<x<\infty). \tag{18b}$$

It is assumed that A may be written as the sine transform of a function B, i.e.

$$A(r) = \frac{2}{\pi}\int_0^\infty \sin rtB(t)\,dt. \tag{19}$$

Equation (19) satisfies (18*a*) provided

$$B(t) = \int_0^t g(x)\,dx \quad (0 < t < 1). \tag{20}$$

This is a direct consequence of substituting (19) into (18*a*). If (19) is substituted into (18*b*) an integral equation is obtained for $B(t)$

$$\frac{1}{\pi}\int_1^\infty B(t)\{K_0(s\,|t-x|)-K_0(s\,|t+x|)\}\,dt = G(x) \quad (1 < x < \infty), \tag{21}$$

where $G(x)$ is known and is given by

$$G(x) = -\frac{1}{\pi}\int_0^1 B(t)\{K_0(s\,|t-x|)-K_0(s\,|t+x|)\,dt \quad (0 < x < \infty). \tag{22}$$

Introduce a sectionally holomorphic function $F(W)$, having a jump discontinuity on the real axis, $1 < |x| < \infty$

$$F(W) = \frac{d}{dW}\int_1^\infty B(t)\{K_0[s(t-W)]-K_0[s(t+W)]\}\,dt. \tag{23}$$

It is necessary that $B(t) \to 0$ as $t \to \infty$ for the existence of the integral (19). This condition is sufficient to determine the asymptotic behaviour of (23). In fact, for $t > R$, $|B(t)| < 1$,

$$\underset{W\to\infty}{F(W)} \leqslant \left|\frac{d}{dW}\int_1^R B(t)\{K_0[s(t-W)]-K_0[s(t+W)]\}\,dt\right| + |K_0[s(t-W)]-K_0[s(t+W)]|. \tag{24}$$

Consider R as a large but fixed arbitrary constant. Then the asymptotic behaviour of $F(W)$, as given by (24), is determined from the behaviour of $K_0(W)$. As $W \to \infty$, $K_0(W) = O(W^{-\frac{1}{2}}\exp Ws)$. Hence for W sufficiently large

$$F(W) = O(W^{-\frac{1}{2}}\exp sW)+O(W^{-\frac{1}{2}}\exp -sW). \tag{25}$$

Set $$F(W) = F_1(W)\exp(-sW)+F_2(W)\exp(+sW). \tag{26}$$

$F_1(W)$ and $F_2(W)$ must be $O(W^{-\frac{1}{2}})$ as $W \to \infty$ for (25) to be satisfied. $F(W)$ as defined in (23) is an even function of W which imposes the further condition

$$F_1(W) = F_2(-W). \tag{27}$$

A functional relationship can now be obtained between $F_1^+(x)$ and $F_1^-(x)$, the values of $F_1(W)$ for $W \to x+i0$ and $W \to x-i0$, respectively.

This functional relationship is obtained by letting W approach the real axis alternatively through positive and negative imaginary values

$$\tfrac{1}{2}[F^+(x)+F^-(x)] = \tfrac{1}{2}[F_1^+(x)+F_1^-(x)]\,e^{-xs}+\tfrac{1}{2}[F_1^+(-x)+F_1^-(-x)]\,e^{xs} = G'(x) \quad (1 < x < \infty); \tag{28a}$$

$$\tfrac{1}{2}[F^+(x)-F^-(x)] = \tfrac{1}{2}[F_1^+(x)-F_1^-(x)]\,e^{-xs}+\tfrac{1}{2}[F_1^+(-x)-F_1^-(-x)]\,e^{+xs} = 0 \quad (0 < x < 1). \tag{28b}$$

Equation (22) implies that as $x \to \infty$, $G'(x) \sim x^{-\frac{1}{2}} e^{-sx}$. Thus in order that the functional equation (28) be valid, it is sufficient that

$$\left.\begin{aligned} \tfrac{1}{2}[F_1^+(x)+F_1^-(x)] &= e^{+xs}G'(x) && (1 < x < \infty), \\ F_1^+(x)+F_1^-(x) &= 0 && (-\infty < x < -1), \\ F_1^+(x)-F_1^-(x) &= 0 && (-1 < x < +1). \end{aligned}\right\} \tag{29}$$

(29) defines a Riemann–Hilbert problem for three disconnected arcs situated along the real axis. This problem is to find a sectionally holomorphic function $F_1(W)$ behaving as $W^{-\frac{1}{2}}$ at infinity and satisfying the boundary conditions stated in (29). The solution to this problem has been treated extensively in the literature (Muskhelishvili 1946) and thus will be stated rather than derived

$$F_1(W) = \frac{(1-W^2)^{\frac{1}{2}}}{W\pi}\int_1^\infty \frac{e^{+st}t}{(t^2-1)^{\frac{1}{2}}}\frac{G'(t)}{(t-W)}\,dt. \tag{30}$$

Hence

$$F(W) = \pi^{-1}W^{-1}(1-W^2)^{\frac{1}{2}}\int_1^\infty \frac{tG'(t)}{(t^2-1)^{\frac{1}{2}}}\left\{\frac{\exp\{s(t-W)\}}{t-W}-\frac{\exp\{s(t+W)\}}{t+W}\right\}dt. \tag{31}$$

The solution of the Riemann–Hilbert problem given by (31) assures continuity of the potential function Φ_e at $x = b$. Since $F(W)$ is known, we can write

$$\begin{aligned} b\alpha\int_0^\infty a_e(q,p)\cos qx\,dq &= \tfrac{1}{2}[F^+(x)+F^-(x)]-\frac{1}{\pi}G'(x) \\ &= \alpha b\Phi_e(x,p,0) \quad (0 < x < b). \end{aligned} \tag{32}$$

Now $F^+(x)+F^-(x)$ can be determined explicitly in terms of (31), so

$$\begin{aligned} &b\alpha\int_0^\infty a_e(q,p)\cos(\alpha qx)\,dq = \alpha b\Phi_e(x,p,0) \\ &= -\frac{(b^2-x^2)^{\frac{1}{2}}}{\pi x}b\alpha\int_b^\infty \frac{tG_e'(t)}{(t^2-b^2)^{\frac{1}{2}}}\left\{\frac{\exp[\alpha p(t-x)]}{t-x}-\frac{\exp[\alpha p(t+x)]}{t+x}\right\}dt+b\alpha G_e'(t), \end{aligned} \tag{33}$$

where $$G_e'(t) = -\frac{1}{\pi}\frac{d}{dt}\int_0^b [K_0(\alpha p\,|t-r|)-K_0(\alpha p\,|t+r|)]\int_0^r \frac{\partial\Phi_e}{\partial z}(x,p,0)\,dx\,dr. \tag{34}$$

The inverse Laplace transform of (33) then yields $\phi_e(x,y,0)$

$$\left.\begin{aligned} \phi_e(x,y,0) &= -\frac{(b^2-x^2)^{\frac{1}{2}}}{\pi x}\int_b^\infty \frac{t}{(t^2-b^2)^{\frac{1}{2}}}\left\{\frac{M(t,x,y)}{t-x}-\frac{M(t,-x,y)}{t+x}\right\}dt \\ &\qquad + M(x,x,y) \quad (0 < |x| < b) \\ &= 0 \quad (b < |x| < \infty), \end{aligned}\right\} \tag{35}$$

where $M(t,x,y) = \mathscr{L}^{-1}[G_e'(t)\exp\{\alpha p(t-x)\}]$ and

$$M(t,x,y) = -\frac{1}{\pi}\frac{d}{dt}\int_{-b}^{+b}\int_0^{y-\alpha(x-t)} \frac{H[y-y'-\alpha\,|(x-r)|]}{\{[y-y'+\alpha(t-x)]^2-\alpha^2(t-r)^2\}^{\frac{1}{2}}} \times\int_0^r \frac{\partial\phi_e}{\partial z}(x',y',0)\,dx'\,dy'\,dr \tag{36}$$

and
$$\left.\begin{aligned} H(x) &= 1 \quad (x > 0), \\ &= \tfrac{1}{2} \quad (x = 0), \\ &= 0 \quad (x < 0). \end{aligned}\right\} \tag{37}$$

Since the potential function $\phi_e(x, y, 0)$ is specified on the plane $z = 0$, the general solution $\phi_e(x, y, z)$ can be obtained from Hadamard's (1923) solution of the partial differential equation (1),

$$\phi_e(x, y, z) = -\frac{1}{\pi} \iint \phi_e(x', y', 0^+) \frac{\partial}{\partial z} [(y-y')^2 - \alpha^2(x-x')^2 - \alpha^2 z^2]^{-\frac{1}{2}} \, dy' \, dx', \tag{38}$$

where the integration is taken over the plane $z = 0$

$$y' \leqslant y - \alpha\{(x-x')^2 + z^2\}^{\frac{1}{2}}. \tag{39}$$

The bars through the integral signs in (38) denote the finite part of this improper integral.

The potential function $\Phi_0(x, p, 0)$ is obtained in a similar manner

$$\Phi_0(x, p, 0) = -\frac{(b^2-x^2)^{\frac{1}{2}}}{\pi x} \int_0^\infty \frac{t G_0(t)}{(t^2-b^2)^{\frac{1}{2}}} \left[\frac{\exp\{p(t-x)\}}{t-x} - \frac{\exp\{p(t+x)\}}{t+x}\right] dt + G_0(t), \tag{40}$$

where
$$G_0(t) = -\frac{1}{\pi} \int_0^b [K_0(p\,|t-r|) - K_0(p\,|t+r|)] \frac{\partial \Phi_0}{\partial z}(r, p, 0)\, dr. \tag{41}$$

The inverse transform of (37) then yields $\phi_0(x, y, 0)$

$$\left.\begin{aligned} \phi_0(x, y, 0) &= -\frac{(b^2-x^2)^{\frac{1}{2}}}{x\pi} \int_0^\infty \frac{t}{(t^2-b^2)^{\frac{1}{2}}} \left[\frac{N(t, x, y)}{t-x} - \frac{N(t, -x, y)}{t+x}\right] dt \\ &\qquad + N(x, y, 0) \quad (0 < |x| < b), \\ &= 0 \quad (b < |x| < \infty), \end{aligned}\right\} \tag{42}$$

where
$$N(t, x, y) = \mathscr{L}^{-1}[G_0(t) \exp\{\alpha p(t-x)\}]$$

and
$$N(t, x, y) = -\frac{1}{\pi} \int_{-b}^{+b} \int_0^{y-\alpha(x-t)} \frac{H[y-y'-\alpha(|x-r|)]}{\{[y-y'+\alpha(t-x)]^2 - \alpha^2(t-r)^2\}^{\frac{1}{2}}} \frac{\partial \phi_0}{\partial z}(r, y', 0)\, dy'\, dr. \tag{43}$$

The function $\phi_0(x, y, z)$ can be expressed in terms of Hadamard's solution of the partial differential equation (1)

$$\phi_0(x, y, z) = -\frac{1}{\pi} \iint \phi_0(x', y', 0) \frac{\partial}{\partial z} [(y-y')^2 - \alpha^2(x-x')^2 - \alpha^2 z^2]^{-\frac{1}{2}} \, dy' \, dx', \tag{44}$$

where the integration is taken over the $z = 0$ plane

$$y' \leqslant y - \alpha\{(x-x')^2 - \alpha z^2\}^{\frac{1}{2}}.$$

Equations (11), (38) and (44) enable us to write $\phi_2(x, y, z)$ explicitly in terms of $\phi_0(x, y, z($ and $\phi_e(x, y, z)$

$$\phi_2(x, y, z) = -\frac{1}{\pi} \iint [\phi_0(x', y', 0) + \phi_e(x', y', 0)] \times \frac{\partial}{\partial z} [(y-y')^2 - \alpha^2(x-x')^2 - \alpha^2 z^2]^{-\frac{1}{2}} \, dy' \, dx'. \tag{45}$$

4. Asymptotic behaviour of $\phi_2(x, y, 0)$ for large distances downstream

In this section we compute the asymptotic behaviour of $\phi(x,y,0)$ for large distances downstream on the aerofoil for the special case of a plane wing of rectangular planform having a semi-span b and at an angle of incidence β with the free stream. This particular case was considered by Stewartson (1950) who assumed an approximate expansion for $\Phi(x, p, 0)$ of the form

$$p\Phi(x,p,0) = \sum_{n=0}^{\infty} a_n(p)\,(1-x^2/b^2)^{n+\frac{1}{2}} \tag{46}$$

for small values of p, and he determined the first few coefficients $a_n(p)$. We shall now show that our exact solution reduces to the first term of the expansion (46) for $p = 0$ which represents the asymptotic behaviour of $\phi(x,y,0)$ as y approaches infinity, i.e.

$$\lim_{y\to\infty} \phi(x,y,0) = \lim_{p\to 0} p\Phi(x,p,0). \tag{47}$$

For the special case of a wing of rectangular planform and at an angle β with the direction of the free stream, $\phi_0(x,y,z)$ and $\phi_1(x,y,z)$ are identically zero, and therefore

$$\phi(x,y,0) = \phi_e(x,y,0). \tag{48}$$

The mean surface of the wing is, to a first approximation, in the plane $z = 0$. The equation of its surface is $|x| < b$, $y > 0$. The boundary conditions in the $z = 0$ plane are

$$\phi = 0 \quad \text{and} \quad \partial\phi/\partial y = 0 \quad \text{for} \quad y \leqslant 0, \tag{49a}$$

$$\phi = 0 \quad \text{on} \quad z = 0 \quad \text{for} \quad |x| < b, \tag{49b}$$

$$\partial\phi/\partial z = -U\beta \quad \text{on the mean surface of the wing, } |x| < b, \quad y > 0. \tag{49c}$$

The Laplace transform of $\partial\phi/\partial z$ with respect to y on the mean surface of the wing is

$$p\,\partial\Phi/\partial z(x,p,0) = -U\beta \quad (|x| < b). \tag{50}$$

The boundary conditions (49) for $\phi(x,y,z)$ also apply to $\phi_e(x,y,z)$ from condition (48).

Substituting (50) into (33) and using (47), it can be shown that

$$\lim_{y\to\infty} \phi(x,y,0) = -\frac{\beta U}{\pi x}\int_0^b t\left[\frac{1}{x-t} - \frac{1}{x+t}\right]dt + \frac{\beta U(b^2-x^2)^{\frac{1}{2}}}{\pi^2 x}\int_b^\infty \frac{2rx(r^2-x^2)^{-\frac{1}{2}}}{r^2-x^2}\int_0^b \frac{2t^2}{r^2-t^2}\,dt\,dr. \tag{51}$$

Interchanging the orders of integration of the t and r integrals and evaluating the resulting integral, one obtains

$$\lim_{y\to\infty} \phi(x,y,0) = -\frac{\beta U}{\pi}(b^2-x^2)^{\frac{1}{2}}\int_0^b t(b^2-t^2)^{-\frac{1}{2}}\left[\frac{1}{t-x} + \frac{1}{t+x}\right]dt. \tag{52}$$

For $|x| < b$, $\phi(x,y,0)$ as defined by the right-hand side of (52) can be evaluated as

$$\lim_{y\to\infty} \phi(x,y,0) = \beta U(b^2-x^2)^{\frac{1}{2}}, \tag{53}$$

which is in agreement with the potential function obtained by Stewartson (1950).

5. Potential function for wings of infinite span ($b = \infty$)

As the span of the wing approaches infinity ($b \to \infty$), the equations for the perturbation potentials ϕ_e and ϕ_0, given by (35) and (42), reduce to

$$\phi_e(x,y,z) = -\frac{1}{\pi}\int_{-\infty}^{+\infty}\int_0^{y-\alpha\{(x-x')^2-z^2\}^{\frac{1}{2}}} \frac{\partial\phi_e}{\partial z}(x',y',0) \times [(y-y')^2-\alpha^2(x-x')^2-\alpha^2z^2]^{-\frac{1}{2}}\,dy'\,dx', \tag{54}$$

$$\phi_0(x,y,z) = -\frac{1}{\pi}\int_{-\infty}^{+\infty}\int_0^{y-\alpha\{(x-x')^2-z^2\}^{\frac{1}{2}}} \frac{\partial\phi_0}{\partial z}(x',y',0) \times [(y-y')^2-\alpha^2(x-x')^2-\alpha^2z^2]^{-\frac{1}{2}}\,dy'\,dx'. \tag{55}$$

Therefore

$$\phi_2(x,y,z) = -\frac{1}{\pi}\int_{-\infty}^{+\infty}\int_0^{y-\alpha\{(x-x')^2-z^2\}^{\frac{1}{2}}} \frac{\partial\phi_2}{\partial z}(x',y',0) \times [(y-y')^2-\alpha^2(x-x')^2-\alpha^2z^2]^{-\frac{1}{2}}\,dy'\,dx'. \tag{56}$$

$\phi_2(x,y,z)$ as given by (56) represents the perturbation potential at the point (x,y,z) due to a distribution of sources in a semi-infinite plane.

6. Extension to more general type leading edges

The above results may be extended to wings of infinite length having partly supersonic and subsonic leadings edges with sides parallel to the free-stream direction. The well-known procedure for finding the normal velocity $\partial\phi/\partial z$ in the vicinity of the leading edge is to transform the Hadamard integral equation

$$\phi(x,y,0) = \frac{1}{\pi}\iint \left.\frac{\partial\phi}{\partial z'}\right|_{z'=0} \frac{H[y-y'-\alpha\,|(x-x')|]}{[(y-y')^2-\alpha^2(x-x')^2]^{\frac{1}{2}}}\,dy'\,dx' \tag{57}$$

to characteristic co-ordinates in the $z = 0$ plane (see Ward 1955), by setting

$$\left.\begin{aligned} \zeta &= y-\alpha x, & \eta &= y+\alpha x, \\ \zeta' &= y'-\alpha x', & \eta' &= y'+\alpha x'. \end{aligned}\right\} \tag{58}$$

Similarly, the equation of the leading edge is expressed in terms of these characteristic co-ordinates. Setting $\phi = 0$ before the aerofoil, one may obtain a simple Abel integral equation (Ward 1955) of the first kind for $\partial\phi/\partial z$ off the aerofoil in terms of $\partial\phi/\partial z$ on the aerofoil. Once knowing $\partial\phi/\partial z$ before the aerofoil, we have reduced the problem to a wing having a straight leading edge which is infinite in length, and the results of the previous sections are applicable.

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